Euler's Method

طريقة اويلر

Consider the first order and first degree differential equation :

$$\frac{dy}{dx} = f(x, y) \qquad \ldots 1$$

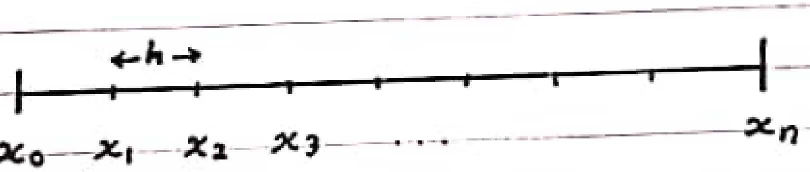
With the condition that $y(x_0) = y_0$

Suppose we want to find the approximate value of y, say y_n when $x = x_n$.

We divide the interval $[x_0, x_n]$ into n subintervals of length, say h with the division points :

$$x_0, x_1, x_2, \ldots \ldots x_n$$

Where $x_r = x_0 + rh \qquad r = 0, 1, 2 \ldots n$



Let us assume that :

$$f(x, y) \simeq f(x_{r-1}, y_{r-1})$$ in the interval $[x_{r-1}, x_r]$

Integrating equation 1 in the interval $[x_{r-1}, x_r]$ we get :

$$\int_{y_{r-1}}^{y_r} dy = \int_{x_{r-1}}^{x_r} f(x,y)\,dx$$

$$\therefore \quad y_r - y_{r-1} = \int_{x_{r-1}}^{x_r} f(x,y)\,dx$$

$$\therefore \quad y_r \approx y_{r-1} + f(x_{r-1}, y_{r-1}) \int_{x_{r-1}}^{x_r} dx$$

$$y_r \approx y_{r-1} + f(x_{r-1}, y_{r-1}) * (x_r - x_{r-1})$$

$$\therefore \quad y_r \approx y_{r-1} + f(x_{r-1}, y_{r-1}) * h$$

The last equation is called Euler's formula.

Taking $r = 1, 2, \ldots n$ in this equation we get the successive approximations of y as follows:

$$y_1 = y_0 + h f(x_0, y_0)$$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$\vdots$$

$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$$

Euler's method has limited usage because of the large error that is accumulated as the process proceeds. The process is very slow and to obtain reasonable accuracy with

Euler's method we have to take a smaller value of h.

Further, the method should not be used for a larger range of x as the values found by this method go on becoming farther and farther away from the true values.

Example: Solve the equation:

$$\frac{dy}{dx} = 1 - y$$

with the initial condition: $x = 0,\ y = 0$ using Euler's algorithm and tabulate the solutions at $x = 0.1, 0.2, 0.3$

Solution:

$$\frac{dy}{dx} = 1 - y \Rightarrow f(x, y) = 1 - y$$

$x = 0.1, 0.2, 0.3 \Rightarrow$

$h = 0.1$

$x_0 = 0$

$y_0 = 0$

$x_1 = x_0 + h = 0 + 0.1$

$= 0.1$

$x_2 = 0.2$

$x_3 = 0.3$

We have $y_{n+1} = y_n + h f(x_n, y_n)$

Take $n = 0$ $\quad y_1 = y_0 + h f(x_0, y_0)$

$= 0 + 0.1 * (1 - 0)$

$= 0.1$

Take $n=1$ $\quad y_2 = y_1 + h\,f(x_1, y_1)$

$$= 0.1 + 0.1 * (1 - y_1)$$
$$= 0.1 + 0.1 * (1 - 0.1)$$
$$= 0.19$$

Take $n=3$ $\quad y_3 = y_2 + h\,f(x_2, y_2)$

$$= 0.19 + 0.1 * (1 - 0.19)$$
$$= 0.271$$

Hence we get :

x	y
0	0
0.1	0.1
0.2	0.19
0.3	0.271

Example :

Given $\dfrac{dy}{dx} = x^3 + y$, $y(0) = 1$

Compute $y(0.02)$ by Euler's method taking $h = 0.01$

solution :

$$\frac{dy}{dx} = x^3 + y \Rightarrow f(x,y) = x^3 + y$$

$x_0 = 0$ $\quad h = 0.01$ $\quad y_0 = 1$

$x_1 = 0.01$

$x_2 = 0.02$

Applying Euler's formula for $n=1$

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$= 1 + (0.01) * (x_0^3 + y_0)$$

$$= 1 + (0.01) * (0^3 + 1)$$

$$= 1.01$$

for $n = 2$

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$= 1.01 + (0.01) * (x_1^3 + y_1)$$

$$= 1.01 + (0.01) * (0.01^3 + 1.01)$$

$$= 1.0201$$

Example :

Solve by Euler's method the following differential equation at $x = 0.1$:

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$ with the initial condition $y(0) = 1$ Taking $h = 0.02$

Solution :

$$f(x, y) = \frac{y-x}{y+x}$$

$x_0 = 0$ $y_0 = 1$ $h = 0.02$

$\therefore x_1 = 0.02 \qquad x_2 = 0.04 \qquad x_3 = 0.06$

$x_4 = 0.08 \qquad x_5 = 0.1$

Using Euler's formula for $n = 1$

$$y_1 = y_0 + h f(x_0, y_0)$$

$$= 1 + 0.02 * \frac{1-0}{1+0}$$

$$= 1.02$$

For $n=2$ $\quad y_2 = y_1 + h f(x_1, y_1)$

$$= 1.02 + 0.02 * \frac{1.02 - 0.02}{1.02 + 0.02}$$

$$= 1.0392$$

For $n=3$ $\quad y_3 = y_2 + h f(x_2, y_2)$

$$= 1.0392 + 0.02 * \frac{1.0392 - 0.04}{1.0392 + 0.04}$$

$$= 1.0577$$

For $n=4$ $\quad y_4 = y_3 + h f(x_3, y_3)$

$$= 1.0577 + 0.02 * \frac{1.0577 - 0.06}{1.0577 + 0.06}$$

$$= 1.0756$$

For $n=5$ $\quad y_5 = y_4 + h f(x_4, y_4)$

$$= 1.0756 + 0.02 * \frac{1.0756 - 0.08}{1.0756 + 0.08}$$

$$= 1.0928$$

Hence we have :

x	y
0	1
0.02	1.02
0.04	1.0392
0.06	1.0577
0.08	1.0756
0.1	1.0928

Home Work :

* Draw the flow chart of Euler's method
* Find $y(1)$ by Euler's method from the differential equation

$$\frac{dy}{dx} = \frac{-y}{1+x} \quad \text{when} \quad y(0.3) = 2$$

taking the step length $h = 0.1$

* Given $\frac{dy}{dx} = x^2 + y$ with $y(0) = 1$ evaluate $y(0.02)$, $y(0.04)$ by Euler's method.
* Given $\frac{dy}{dx} = y - x$ where $y(0) = 2$ find $y(0.1)$ and $y(0.2)$ by Euler's method.
* Given $\frac{dy}{dx} = \frac{y-x}{1+x}$ with boundary condition $y(0) = 1$, find y for $x = 0.1$ by Euler's method (five steps).

* Solve $y' = x - y^2$ by Euler's method for $x = 0.2$ to 0.6 with $h = 0.2$ initially $x = 0$, $y = 1$

* Using Euler's method solve the equation $\frac{dy}{dx} = x + |\sqrt{y}|$ with boundary condition $y = 1$ at $x = 0$ for the range $0 \leq x \leq 0.4$ in steps of 0.2.